M.Sc Sem I

Page.No.:
Date: / /

A group G is said to be solvable if we can find a finite chain of subgroup

$$G = N_0 \supset N_1 \supset N_2 \supset \dots \supset N_K = \{e\}$$

s.t. each N_{i+1} is a normal subgroup of N_i and each quotient group N_i / N_{i+1} is abelian. Then the above series is referred to as solvable series of G.

Theorem based Solvable Group

Th. 1 A group G is solvable iff $G^{(n)} = \{e\}$

Proof: Let G be group. Then $G^{(n+1)} = (G^{(n)})'$ is a normal subgroup of $G^{(n)}$ $\forall n$.

Write $N_K = G^{(K)}$. Then N_{K+1} is a normal subgroup of N_K $\forall K$

step Let $G^{(n)} = \{e\}$

To prove that G is solvable.

Scanned with CamScanner

Page No.:
Date: / /

Write
$N_0 = G$, $N_1 = G^{(1)}$, $N_2 = G^{(2)}$
$= (G')'$, ... $N_K = G^{(K)}$.

N_{K+1} is a normal subgroup of N_K

$\Rightarrow G = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_n = \{e\}$ — (1)

Also
$$\frac{N_K}{N_{K+1}} = \frac{G^{(K)}}{G^{(K+1)}} = \frac{G^K}{(G^K)'}$$

$G^{(K)} / G^{(K)'}$ is abelian

Consequently N_K / N_{K+1} is abelian group $\forall K$. Hence (1) is a solvable series for G. Consequently G is a solvable group.

Step II. Let G be a solvable group, so that $\exists$ a solvable series

Page No.:
Date: / /

for G namely

$$G = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_n = \{e\} \quad \text{— (2)}$$

To prove that
$G^{(n)} = \{e\}$

By definition of soluable series N_{K+1} is a normal subgroup of N_K $\forall K$ and

N_K / N_{K+1} is an abelian quotient group $\forall K$

To prove that $G^{(K)} = \{e\}$

Our assumption $\Rightarrow$ the commutator subgroup N'_{K+1} of N_K must be contained in N_K — (3)

$$\Rightarrow N'_{K+1} \subset N_K \quad \text{— (3)}$$

$$\Rightarrow N_0' \subset N_1 \Rightarrow G' \subset N_1 \quad (K=1)$$

and

$$N_1' \subset N_2 \Rightarrow N_2 \supset N_1' \supset (G')' = G^{(2)}$$

$K = 2$

Page No.:
Date: / /

[For $G' \subset N_1 \Rightarrow (G')' \subset (N_1)'$]

$\Rightarrow N_2 \supset G^{(2)}$

$N_2' \subset N_3 \Rightarrow N_3 \supset N_2' \supset (G^{(2)})'$
$= G^{(3)}$

$\Rightarrow N_3 \supset G^{(3)}$

Generalising this $N_n \supset G^{(n)}$

By 2 $N_n = \{e\}$ Hence

$\{e\} \supset G^{(n)}$

But $\{e\} \subset G^{(n)}$ is always true

Consequenly $G^{(n)} = e$